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SYMMETRIC DESIGNS AND GEOMETROIDS

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A λ-set S in a symmetric $2-(v, k, \lambda)$ design Π is a subset which every block meets in 0, 1 or λ points such that for any point of S there is a unique block meeting S at that point only. Ovoids in three-dimensional projective spaces are examples of λ-sets. It is shown that if Π has a λ-set then Π is a geometroid with $v=\lambda u^2+u+1$ and $k=\lambda u+1$, where $u \geqq \lambda-1$. The cases when u is $\lambda-1$, λ and $\lambda+1$ are investigated and some open problems discussed.

1. Introduction

The motivation for this work came from an attempt to classify families of symmetric 2-designs. Many such families have been found in recent years by a variety of methods. The more general aim of this paper was to try a classification of some families by looking at those containing certain subsystems, called here λ-sets. It turns out that the families containing λ-sets are geometroids.

In the case of the design of the points and hyperplanes of a finite three-dimensional projective space, a λ-set is an ovoid. More generally, λ-sets are special cases of what Sane et al. [8] call a (λ, s)-arc in a symmetric 2-design and what Calderbank and Kantor [2] call projective (n, k, h_1, h_2) sets in projective spaces.

After looking at general symmetric 2-designs which have λ-sets we consider in turn three families of geometroids and discuss the existence of designs having λ-sets in each family, and some open problems which arise from this.

For the basic design and geometric definitions and results used in this paper see Hughes and Piper [5] or Dembowski [3].

2. Parameters

Throughout this section Π denotes a symmetric $2-(v, k, \lambda)$ design which has a subset S of s points such that:

(i) $\lambda \geqq 2$, $s \geqq 1$;

(ii) $|S \cap B| \in \{0, 1, \lambda\}$ for any block B;

(iii) if $p \in S$, there is a unique block B such that $B \cap S=\{p\}$.

Such a subset S of the design Π is called a *λ-set.*

Note that the case $\lambda=1$ has been excluded as it will not be of interest in the discussion which follows, for reasons which will become apparent.

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Relative to the λ-set S, a block B will be called an *internal block, tangent block* or *secant block* if, respectively, it meets S in 0, 1 or λ points. It is clear from the definition that S has exactly s tangents.

Let the number of internal and secant blocks of S be α and β respectively. Then clearly $\alpha+\beta=v-s$. It is also clear that points of S together with the secant blocks form a $2-(s, \lambda, \lambda)$ design with β blocks and "r"$=k-1$.

Denote this $2-(s, \lambda, \lambda)$ design by Π_S.

The basic design parameter equations easily give $\lambda\beta=s(k-1)$ and $(k-1)(\lambda-1)=\lambda(s-1)$. Then using $\alpha+\beta=v-s$ and simplifying we find that:

$$(2.1) \qquad s = 1+(\lambda-1)(k-1)/\lambda = (\lambda k+1-k)/\lambda$$

$$\alpha = (k-1)(k+\lambda-\lambda^2-1)/\lambda^2$$

$$\beta = (k-1)(\lambda k+1-k)/\lambda^2.$$

It therefore follows that λ divides $k-1$ and hence that Π is a geometroid $G_u(\lambda)$ in the sense of Mullin [6], where $u=(k-1)/\lambda$.

It is also worth observing here that, in the terminology of Sane et al [8], S is a (λ, s)-arc; also that an arc of Assmus and van Lint [1] is a λ-set if and only if $\lambda=2$.

(2.2) Lemma. *Let p be a point of Π not in S and let α', β', γ' be, respectively, the number of internal, secant and tangent blocks on p. Then γ' is either 0 or λ.*

If $\gamma'=0$, then $\alpha'=(k-1)/\lambda$ and $\beta'=s=(\lambda k+1-k)/\lambda$.

If $\gamma'=\lambda$, then $\alpha'=(k+\lambda-\lambda^2-1)/\lambda$ and $\beta'=s-1=(\lambda-1)(k-1)/\lambda$.

Proof. Clearly $\alpha'+\beta'+\gamma'=k$. Counting ordered pairs (q, B), where $p, q\in B$, $q\in S$, gives easily $\lambda\beta'+\gamma'=\lambda s=\lambda k+1-k$. Now since λ divides $k-1$, then λ divides γ'. Hence either $\gamma'=0$ or else $\gamma'\geqq\lambda$.

Letting p vary over all points not in S, we get the following sums:

$$\sum \gamma' = s(k-1)$$

and

$$\sum \gamma'(\gamma'-1) = s(s-1)\lambda$$

from which, using (2.1), it is easily deduced that $\sum \gamma'(\gamma'-\lambda)=0$. Hence since for each p either $\gamma'=0$ or $\gamma'\geqq\lambda$, it follows that if $\gamma'\neq 0$ then necessarily $\gamma'=\lambda$.

Finally, using the above equations and (2.1) the proof is readily completed. ∎

We have shown in the above lemma that any point not on the λ-set S is on either 0 or λ tangent blocks.

Definition. A point p of Π is called an *internal point* or *secant point*, relative to S, according as the number of tangent blocks on p is 0 or λ.

The following lemma follows easily using the previous lemma.

(2.3) Lemma. *In the dual symmetric design of Π, the tangent blocks of S form a λ-set whose tangent blocks are the points of S and whose internal (secant) blocks are the internal (secant) points of S.* ∎

Next we investigate the type of parameters, in certain cases, that Π may have and discuss the existence of such a design having a λ-set.

Let $u=(k-1)/\lambda$. Then Π is a symmetric $2-(\lambda u^2+u+1, \lambda u+1, \lambda)$ design and $s=\lambda u-u+1$, $\alpha=u(u+1-\lambda)$, $\beta=u(\lambda u-u+1)=us$. As mentioned earlier, Π has the parameters of a geometroid $G_u(\lambda)$.

Rajkundlia [7, p. 82] constructs examples of $G_u(\lambda)$ when both u and $\lambda u+1$ are prime powers. See also Shrikhande and Singhi [9] for the case $\lambda=u$.

Our interest here is not just with parameters but with the existence of geometroids with λ-sets. Since $\alpha\geqq 0$ it follows that $\lambda-1\leqq u$. We shall proceed to examine, in turn, the parameter types when u is $\lambda-1$, λ and $\lambda+1$.

Type I: $u=\lambda-1$.

Here Π is a $2-(\lambda^3-2\lambda^2+2\lambda, \lambda^2-\lambda+1, \lambda)$ design with $s=\lambda^2-2\lambda+2$, $\alpha=0$ $\beta=(\lambda-1)(\lambda^2-2\lambda+2)$. Thus every block meets S in either 1 or λ points.

In this case, the design Π_S is a $2-(\lambda^2-2\lambda+2, \lambda, \lambda)$ design. It is interesting to note that any inversive plane of order $\lambda-1$ will have these parameters as a 2-design. Inversive planes of order n are known to exist when n is a prime power (see, for example, [3] or [5]).

Non-degenerate quadrics of index one in the projective geometry $PG(3, q)$ are examples of λ-sets. A hyperplane meeting such a quadric in one point is the tangent hyperplane at that point. The points and hyperplanes of $PG(3, q)$ form a symmetric $2-(q^3+q^2+q+1, q^2+q+1, q+1)$ design which is of Type I with $\lambda=q+1$. Denote the design by $P(3, q)$.

By a theorem of Calderbank and Kantor [2, Theorem 12.6], a λ-set in $P(3, q)$ must be an ovoid and, if q is odd, the ovoid is a quadric in $PG(3, q)$. (See Dembowski [3] for information on ovoids.) Ebert [4, Theorem 3] has proved that the points of $P(3, q)$, $q>2$, admit a partition by ovoids.

Observe that the designs Π_S in the $P(3, q)$ example are all inversive planes and hence 3-designs. We do not know of any other Type I designs. An interesting question is whether a Type I design which admits a partition by λ-sets is necessarily isomorphic to some $P(3, q)$ or, if not, then need the designs Π_S be 3-designs?

Type II: $u=\lambda$.

Π is now a symmetric $2-(\lambda^3+\lambda+1, \lambda^2+1, \lambda)$ design with $s=\lambda^2-\lambda+1$, $\alpha=\lambda$, $\beta=\lambda(\lambda^2-\lambda+1)$ and Π_S is a $2-(\lambda^2-\lambda+1, \lambda, \lambda)$ design. Note that Π_S has the parameters of the sum of λ projective planes of order $\lambda-1$.

Let p be any point of Π not in S. Then, with the notation of (2.2), we have that either: (i) $\alpha'=\lambda$, $\beta'=\lambda^2-\lambda+1$, $\gamma'=0$ or (ii) $\alpha'=1$, $\beta'=\lambda^2-\lambda$, $\gamma'=\lambda$. In (i), p is an internal point and in (ii) it is a secant point.

From the dual of (2.2) it follows that any internal block is on exactly λ internal points and therefore contains all the internal points.

Designs with Type II parameters are constructed by Shrikhande and Singhi [9] and also by Rajkundlia [7, p. 84] when both $\lambda-1$ and $\lambda^2-\lambda+1$ are prime powers. It is not difficult to see that the designs in [7] have λ-sets S for which Π_S is a sum of projective planes of order $\lambda-1$.

Here we shall describe a construction, under a hypothesis similar to those above, by first constructing an interesting self-dual 1-design with special block intersection properties, whose automorphism group is transitive. This design is then extended to a symmetric 2-design. Our construction of the symmetric 2-designs will show how they are composed from other designs and that they are self-dual.

The method requires $\lambda-1$ to be the order of a projective plane and $\lambda^2-\lambda+1$ must be a prime power. Whether or not our construction gives designs isomorphic to those of [7] or [9] is unclear. Moreover the question of self-duality is not pursued in these papers.

First we describe the construction of the 1-design in a more general form than will be required.

(2.4) Theorem. *Let q be a prime power and t, n positive integers such that $q-1=tn$. Then there exist a symmetric $1-(qt, q-1, q-1)$ design Γ with the following properties.*

(i) *Γ is self-dual and its automorphism group is transitive.*

(ii) *Any two distinct blocks meet in either 0, $n-1$ or n points.*

(iii) *The blocks of Γ may be partitioned into q subsets each consisting of t disjoint blocks.*

(iv) *There is a partition of the blocks of Γ into t subsets of size q such that any pair of blocks from the same subset meet in $n-1$ points.*

Proof. Let H be the multiplicative group and A the additive group of $GF(q)$. Let K be the unique subgroup of H of order n and let L be the quotient group H/K of order t. Choose any $a\in H$.

Then the points of Γ are the pairs (x, y), where $x\in A$, $y\in L$. Given $v\in A$, $w\in L$, a block, denoted by $[v, w]$ is defined to be the following point subset of Γ:

$$\{(x, y)|a(x-v)\in y^{-1}w\}.$$

The mapping $(x, y)\to[-x, y^{-1}]$ is easily seen to induce an isomorphism from Γ onto its dual. Hence Γ is self-dual.

If $e\in A$ and $f\in L$, it is readily verified that the mapping $(x, y)\to(x+e, fy)$ induces an automorphism of Γ and that these automorphisms form a group which is transitive on points (and on blocks). This proves (i).

Consider a block $[v, w]$. A point $(x, y)\in[v, w]$ if and only if $a(x-v)\in y^{-1}w$. The latter condition implies $x\neq v$. Choose any $x\in A$, $x\neq v$. Then there is a unique $y\in L$ such that $a(x-v)\in y^{-1}w$. So $[v, w]$ is on exactly $|A|-1=q-1$ points. Dually every point is on $q-1$ blocks. Thus Γ is a design with the required parameters.

Let $v\in A$ and consider the t blocks $[v, w]$, $w\in L$. If $w_1, w_2\in L$ and (x, y) is in both $[v, w_1]$ and $[v, w_2]$, then $a(x-v)$ is in both the cosets $y^{-1}w_1$, $y^{-1}w_2$; so these cosets are equal and hence $w_1=w_2$. This proves (iii).

Now consider blocks $[v, w]$, $[v', w']$ where $v\neq v'$. We show that they meet in $n-1$ points if $w=w'$ and in n points otherwise. Using the properties of the above automorphism, it is clear that we may assume $[v', w']=[0, 1]$, where $v\neq0$ (since $v\neq v'$).

A point (x, y) is in both $[v, w]$ and $[0, 1]$ if and only if $a(x-v)$ is in $y^{-1}w$ and ax is in y^{-1}; that is, ax is in y^{-1} (implying $x\neq0$) and $1-x^{-1}v$ is in w. Now if $h\in w$, then $1-x^{-1}v=h$, where $x\neq0$, if and only if $h\neq1$ and $x=v(1-h)^{-1}$.

Thus the number of points in which the given blocks meet is the number of elements $\neq1$ in the coset w, which is $|K|-1=n-1$ if $1\in w$ (that is, $w=1$) and is n otherwise.

Thus, given $w\in L$, the q blocks $[v, w]$, $v\in A$, form a subset with the properties given in (iv). This proves (ii) and (iv). ∎

Now we show how a symmetric 2-design may be constructed using (2.4).

(2.5) Theorem. *If* $\lambda^2-\lambda+1$ *is a prime power and there exists a projective plane of order* $\lambda-1$, *then there exists a symmetric* $2-(\lambda^3+\lambda+1, \lambda^2+1, \lambda)$ *design which is self-dual and has a* λ*-set.*

Proof. Using (2.4) and its notation construct a $1-(\lambda^3-\lambda^2+\lambda, \lambda^2-\lambda, \lambda^2-\lambda)$ design Γ taking $q=\lambda^2-\lambda+1=|A|$, $t=\lambda=|L|$ and $n=\lambda-1$.

We extend Γ to an incidence structure Π by adjoining new points and blocks and extending blocks of Γ.

Let $\sum$ be a projective plane of order $\lambda-1$. Let the point set and line set of $\sum$ be, respectively, $\{p_x|x\in A\}$ and $\{B_x|x\in A\}$. The points of Π are those of Γ and the new points labelled (z), where $z\in A\cup L$. The blocks of Π are those of Γ and the new blocks labelled $[z]$, $z\in A\cup L$. Incidences in Π are defined as follows:

(i) $(x, y)\in[v, w]$ if $a(x-v)\in y^{-1}w$. That is, incidences are inherited from Γ. $(x, v\in A;\ y, w\in L.)$

(ii) $(x, y)\in[v]$ if $p_v\in B_x$ in $\sum$. $(x, v\in A;\ y\in L.)$

(iii) $(x, y)\in[y]$. $(x\in A;\ y\in L.)$

(iv) $(x)\in[v, w]$ if $p_x\in B_v$ in $\sum$. $(x, v\in A;\ w\in L.)$

(v) $(x)\in[x]$. $(x\in A\cup L.)$

(vi) $(y)\in[w]$. $(y, w\in L.)$

(vii) $(y)\in[x, y]$. $(x\in A;\ y\in L.)$

Evidently Π has $(\lambda^3-\lambda^2+\lambda)+(\lambda^2-\lambda+1)+\lambda=\lambda^2+\lambda+1$ points and the same number of blocks. Next we show that every block is on exactly λ^2+1 points.

Let $v\in A$, $w\in L$. The block $[v, w]$ is on $\lambda^2-\lambda$ points of Γ and contains (w) and also (x), where x is any of the elements $x\in A$ such that p_x is on B_v in $\sum$; a total of λ^2+1 points.

The block $[v]$, $v\in A$, is on the $\lambda|L|=\lambda^2$ points (x, y), where $y\in L$ and x is such that $p_v\in B_x$ in $\sum$; and also on the point (v). Again a total of λ^2+1 points.

Next we show that any two distinct blocks meet in λ points. Consider two distinct blocks $[v, w]$, $[v', w']$.

Case 1: $(v=v')$. The two blocks meet in the λ points (x), where $p_x\in B_v$ in $\sum$.

Case 2: $(v\neq v',\ w=w')$. The given blocks meet now in the following points: (x), $x\in A$, where p_x is the intersection of the lines B_v and $B_{v'}$; the point (w); the $n-1=$ $=\lambda-2$ points in which they meet in Γ (see proof of (2.4)). They meet therefore in λ points.

Case 3: $(v\neq v',\ w\neq w')$. The points in which the two blocks meet are now: the $n=\lambda-1$ points in which they meet in Γ; the block (x), where p_x is the intersection of the lines B_v, $B_{v'}$ in $\sum$. This again makes a total of λ points.

Next consider the blocks $[v, w]$, $[z]$, where $z\in A$. A point $(x, y)\in[z]$ if an only if $p_z\in B_x$ in $\sum$. The point $(x, y)\in[v, w]$ if and only if $a(x-v)=y^{-1}w$. Now given $x\in A$, $x\neq v$, the latter equation may be solved uniquely for y.

Thus, if $p_z\notin B_v$ in $\sum$, then the λ elements x with $p_z\in B_x$ determine λ points $(x, y)\in[v, w]\cap[z]$; but if $p_z\in B_v$, then the $\lambda-1$ elements x $(\neq v)$ with $p_z\in B_x$ determine $\lambda-1$ such points and in this case point (z) is also in the intersection. Thus the two blocks meet in λ points.

The remainder of the proof that any two blocks meet in λ points is straightforward and is omitted.

It follows that Π is a symmetric design with the required parameters. It is easily checked that the subset $S=\{(x)|x\in A\}$ is a λ-set in Π.

It remains to prove that Π is self-dual. We shall exhibit the isomorphism from Π onto its dual design but omit the verification which is straightforward:

$$(x,y)\to[x,-y^{-1}],\quad (x)\to[x],\quad (y)\to[y^{-1}],$$

$$[v,w]\to(v,w^{-1}),\quad [x]\to(x),\quad [y]\to(-y^{-1}).\quad \blacksquare$$

Remark. It is easy to see that Π_S in (2.5) is just the sum of λ copies of $\sum$. We know of no examples of a Type II design Π with a λ-set S for which Π_S is not a sum of projective planes.

Type III: $u=\lambda+1$.

Here Π is a symmetric $2-(\lambda^3+2\lambda^2+2\lambda+2,\lambda^2+\lambda+1,\lambda)$ design with $s=\lambda^2$, $\alpha=2(\lambda+1)$, $\beta=\lambda^2(\lambda+1)$ and Π_S is a $2-(\lambda^2,\lambda,\lambda)$ design. Note that Π_S has the parameters of a sum of λ affine planes of order λ.

Applying the Bruck—Ryser—Chowla theorem (see, for example, [3] or [5]) it follows that if λ is even, then $k-\lambda=\lambda^2+1$ must be a square, which is impossible.

Hence λ must be odd. If $\lambda=1$, we have the projective plane of order 2 as a unique example of Π. In this case any set consisting of just one point is a λ-set.

More generally, for λ odd, the Bruck—Ryser—Chowla theorem asserts that if Π exists then the following equation has a non-trivial solution in integers:

$$x^2=(\lambda^2+1)y^2+\lambda(-1)^{\mu}z^2 \tag{2.6}$$

where $\mu=\frac{1}{2}(\lambda+1)$.

We know of no design with Type III parameters other than the projective plane of order 2. In such designs, we have observed that λ must be odd. By considering equation (2.6) modulo 8, we see that $\lambda\equiv\pm1 \pmod 8$ and hence infinitely many odd values of λ are excluded. However, if λ is a perfect square, say $\lambda=m^2$, then $x=1$, $y=1$, $z=m$ is a non-trivial solution of (2.6) and so infinitely many values of λ are not excluded by (2.6). A solution of (2.6) with smallest $\lambda(\neq1)$ is $x=15$, $y=1$, $z=5$ and $\lambda=7$.

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